If G and H are groups, we say $G \equiv_{\infty\omega} H$, G and H are $L_{\infty\omega}$ - equivalent, if and only if G and H satisfy the same sentences of $L_{\infty\omega}$. This relationship is clearly stronger than elementary equivalence, but weaker than isomorphism, as will become clear later. G and H do, however, share isomorphisms on subgroups in the following sense.

We say that $\underline{G \cong_p H}$, G is partially isomorphic to H, if there is a nonempty set I of isomorphisms of subgroups of G with subgroups of H with the back-and-forth property: For any $f \in I$ and $a \in G$ (respectively $b \in H$), there is a $g \in I$ such that $f \subset g$ and $a \in dom(g)$ (respectively $b \in rng(g)$). We will sometimes write $I : G \cong_p H$.

Theorem 2.1. *If G and H are countable and $G \cong_p H$ then $G \cong H$.*

The proof is in Barwise [**?**] (Theorem 2) and simply involves starting with any $f \in I : G \cong_p H$, extending it first to an element of G, then to an element of H, then to another element of G, and so on, going back and forth between G and H.

Thus far we have defined two concepts, the logical concept of equivalence in the language $L_{\infty\omega}$ and the algebraic concept of partial isomorphism. Karp's Theorem allows us to use them interchangeably.

Theorem 2.2. *(Karp) $G \equiv_{\infty\omega}$ if and only if $G \cong_p H$.*

The proof is in Barwise [**?**] (Theorem 3), and also shows that if $G \cong_p H$, we may choose $I : G \cong_p H$ such that every $f \in I$ has finitely generated domain and range.

Examples

(1) Any two dense linearly ordered sets without end points are partially isomorphic. In particular, any two countable dense linearly ordered sets without endpoints are isomorphic.
(2) $\prod_{i\in I} G_i \cong_p \sum_{i \in I} G_i$ for any set I and set of infinite cyclic groups $\{G_i\}_{i\in I}$.

These examples show that partially isomorphic groups need not have the same cardinality. For more information on $L_{\infty\omega}$ and partial isomorphisms, including proofs of the above examples, the reader is referred to Barwise's article [**?**].

3. A Generalization of Ulm's Theorem in $L_{\infty\omega}$

In 1933 Ulm [**?**] defined a set of invariants that classify countable torsion groups up to isomorphism. In order to state Ulm's theorem, we must first look at the important concept of height.

In the following definitions, we fix a group G and a prime p. Let $\underline{pG} = \{px : x \in G\}$.

If α is an ordinal, we define $\underline{p^\alpha G}$ by induction on α as follows: $p^{\alpha+1} = p(p^\beta G)$ if $\alpha = \beta+1$ and $p^\alpha G = \bigcap_{\beta<\alpha} p^\beta G$ if α is a limit ordinal. Now we define the p-height of x, $\underline{h_p(x)}$ for $x \in G$, to be the unique ordinal α such that $x \in p^\alpha G$ and $x \notin p^{\alpha+1} G$ if it exists, and the symbol ∞ otherwise. Note that the value of $h_p(x)$ is dependent on the group G, but the group under consideration will be clear from the context. We may talk about the height of x and write $h(x)$ if p is understood.

Examples

(1) If $x \in \mathbb{Z}$, $h_p(x) =$ the exponent of p in the prime factorization of x.